5

Number Plane
Graphs and
Coordinate Geometry

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Maths Terms, Diagnostic Test, Revision Assignment, Working Mathematically

Learning Outcomes

PAS5·1·2 Determines the midpoint, length and gradient of an interval joining two points on the number plane and graphs linear and simple non-linear relationships from equations.
PAS5·2·4 Draws and interprets graphs including simple parabolas and hyperbolas.
PAS5·3·4 Draws and interprets a variety of graphs including parabolas, cubics, exponentials and circles and applies coordinate geometry techniques to solve problems.

Working Mathematically Stages 5·3·1–5
1 Questioning, 2 Applying Strategies, 3 Communicating, 4 Reasoning, 5 Reflecting
5:01 The Parabola

Outcomes PAS5·1·2, PAS5·2·4, PAS5·3·4

• The shape of the parabola is clearly demonstrated by the water arcs of this fountain.

Up until this point, all the graphs have been straight lines. In this section, we will look at a most famous mathematical curve, the parabola.

• The equations of parabolas are called quadratic equations and have \( x^2 \) as the highest power of \( x \).

• The simplest equation of a parabola is:

\[
y = x^2
\]

• As with the straight line, the equation is used to find the points on the curve. Some of these are shown in the table.

\[
\begin{array}{cccccc}
 x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
y & 9 & 4 & 1 & 0 & 1 & 4 & 9 \\
\end{array}
\]

For an accurate graph, many points would have to be plotted.

• From the graph we can see that the parabola has a turning point, or vertex, which is the minimum value of \( y \) on \( y = x^2 \).

• The \( y \)-axis is an axis of symmetry of the curve, so the right side of the curve is a reflection of the left side. This can be seen when points on either side of the axis are compared.

• The parabola is concave up, which means it opens out upwards.

Parabolas can be happy (up) or sad (down).
Exercise 5:01

1. Complete the following tables and then graph all four curves on one number plane.
   *Hint:* On the y-axis, use values from 0 to 12.
   
   **a** \( y = x^2 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   y & & & & & & & \\
   \hline
   \end{array}
   \]

   **b** \( y = 2x^2 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -2 & -1 & 0 & 0.5 & 1 & 2 \\
   y & & & & & & \\
   \hline
   \end{array}
   \]

   **c** \( y = 3x^2 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -2 & -1 & -0.5 & 0 & 0.5 & 1 & 2 \\
   y & & & & & & & \\
   \hline
   \end{array}
   \]

   **d** \( y = 0.5x^2 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   y & & & & & & & \\
   \hline
   \end{array}
   \]

   For the equation \( y = ax^2 \), what is the effect on the graph of varying the value of \( a \)?

2. Match each of the parabolas A to D with the equations below.
   
   **a** \( y = 0.25x^2 \)
   **b** \( y = 5x^2 \)
   **c** \( y = 2x^2 \)
   **d** \( y = 0.1x^2 \)

   These parabolas are all concave up.

3. Complete the following tables and then graph all four curves on one number plane.
   *Hint:* On the y-axis, use values from -2 to 13.
   
   **a** \( y = x^2 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   y & & & & & & & \\
   \hline
   \end{array}
   \]

   **b** \( y = x^2 + 2 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   y & & & & & & & \\
   \hline
   \end{array}
   \]

   **c** \( y = x^2 + 4 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   y & & & & & & & \\
   \hline
   \end{array}
   \]

   **d** \( y = x^2 - 2 \)
   
   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   y & & & & & & & \\
   \hline
   \end{array}
   \]

   What is the difference in the curves \( y = x^2 \) and \( y = x^2 + 2 \)?

   Can you see that the shape of the curve is the same in each case?

   For the equation \( y = x^2 + c \), what is the effect on the graph of varying the value of \( c \)?
4  a  Complete the table of values for \( y = -x^2 \) and sketch its graph.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
x  & -3 & -2 & -1 & -0.5 & 0 & 0.5 & 1 & 2 & 3 \\
\hline
y  &   &   &   &   &   &   &   &   &   \\
\hline
\end{array}
\]

b  Sketch the graph of \( y = -2x^2 \).

c  For \( y = ax^2 \), what does the graph look like if the value of \( a \) is negative?

5  On the same number plane, sketch the graphs of \( y = -x^2 - 1 \) and \( y = -x^2 + 2 \).

6  Match each of the following equations with the graphs A to E.

a  \( y = -x^2 - 2 \)

b  \( y = x^2 + 2 \)

c  \( y = x^2 - 2 \)

d  \( y = x^2 - 4 \)

e  \( y = 4 - x^2 \)

7  The graphs of \( y = x^2 \) and \( y = (x - 2)^2 \) are shown on the diagram.

a  How are the two graphs related?

b  Graph the parabola \( y = (x + 2)^2 \). How is it related to the graph of \( y = x^2 \)?

c  Sketch the graphs of \( y = (x - 4)^2 \) and \( y = (x + 1)^2 \) on different number planes.

d  How are the graphs of \( y = (x - h)^2 \) and \( y = (x + h)^2 \) related to the graph of \( y = x^2 \)?
The graphs of \( y = (x - 3)^2 \) and \( y = (x - 3)^2 + 2 \) are shown on the diagram.

a. How is the graph of \( y = (x - 3)^2 + 2 \) related to the graph of \( y = (x - 3)^2 \)?

b. How would the graph of \( y = (x - 3)^2 - 2 \) be obtained from the graph of \( y = (x - 3)^2 \)?

c. Sketch the graph of \( y = (x + 3)^2 + 2 \).

d. Use the questions in a to c to explain the connection between the equation of the parabola and the coordinates of its vertex.

e. Sketch the graph of each parabola on a separate number plane.
   - i. \( y = (x - 2)^2 + 3 \)
   - ii. \( y = (x + 2)^2 + 3 \)
   - iii. \( y = (x - 4)^2 - 2 \)
   - iv. \( y = (x + 1)^2 - 2 \)

On separate number planes, sketch the following parabolas:

a. \( y = (x + 4)^2 \)

b. \( y = -(x + 4)^2 \)

c. \( y = -(x + 4)^2 + 3 \)

Find the equation of the parabola that results from performing the following transformations on the parabola \( y = x^2 \).

a. moving it up 2 units

b. moving it down 2 units

c. moving it 2 units to the right

d. moving it 2 units to the left

e. turning it upside down and then moving it up 4 units

f. turning it upside down and then moving it down 2 units

g. moving it up 2 units and then reflecting it in the x-axis

h. moving it 2 units to the right and then turning it upside down

i. moving it up 2 units and then moving it 2 units to the left.

j. turning it upside down, moving it 3 units to the left and then moving it down 2 units.

Match each equation with the corresponding parabola in the diagram.

\[ \begin{align*}
   a & \text{ } y = 2x^2 \\
   b & \text{ } y = -\frac{1}{2}x^2 \\
   c & \text{ } y = x^2 + 1 \\
   d & \text{ } y = x^2 + 3 \\
   e & \text{ } y = -x^2 - 1 \\
   f & \text{ } y = 1 - x^2 
\end{align*} \]
Match each equation with the corresponding parabola A to F in the diagram.

a \( y = (x - 3)^2 \)
b \( y = (x + 3)^2 \)
c \( y = -(x + 2)^2 + 4 \)
d \( y = -(x - 2)^2 + 4 \)
e \( y = (x + 3)^2 - 2 \)
f \( y = (x - 3)^2 - 2 \)

The parabolas shown are the result of translating and/or reflecting the parabola \( y = x^2 \). Find the equation of each parabola.
Investigation 5:01  The graph of parabolas

A graphics calculator or computer graphing package are excellent tools for investigating the relationship between the equation of a parabola and its graph.

Use either of the above to investigate graphs of the following forms for varying values of $a$, $h$ and $k$.

1. $y = ax^2$
2. $y = ax^2 \pm k$
3. $y = (x \pm h)^2$
4. $y = (x \pm h)^2 + k$

Write a report on each of the forms, explaining how the features of the graph, such as the concavity, the position of the vertex and the number of $x$-intercepts, are related to the values of $a$, $h$ and $k$.

• The dish of a radio-telescope is parabolic in shape.
5:02 Parabolas of the Form $y = ax^2 + bx + c$

In the last section, we saw that all parabolas have the same basic shape.
- They are all concave up or concave down with a single vertex or turning point.
- They are symmetrical about an axis of symmetry.

We looked at the connection between the parabola’s shape and its equation and at what numbers in the equation influenced the steepness, the concavity and the position of the graph on the number plane.

In this section, we look at how to sketch the parabola when its equation is given in the form $y = ax^2 + bx + c$. We will also look at how to find features of the parabola, such as the $x$- and $y$-intercepts, the axis of symmetry, the vertex and the maximum or minimum value of $y$.

**Finding the $y$-intercept**

To find the $y$-intercept of $y = x^2 + x - 12$, we let $x$ be zero.

$y = x^2 + x - 12$

When $x = 0$, $y = -12$

∴ The $y$-intercept is $-12$.

∴ The curve cuts the $y$-axis at $(0, -12)$.

**Finding the $x$-intercepts**

To find the $x$-intercepts of $y = x^2 + x - 12$, we let $y$ be zero.

$y = x^2 + x - 12$

When $y = 0$, $0 = x^2 + x - 12$

Solving this, $0 = (x + 4)(x - 3)$

∴ $x = -4$ or $3$

∴ The $x$-intercepts are $-4$ and $3$.

∴ The curve cuts the $x$-axis at $(-4, 0)$ and $(3, 0)$.

**Note:** If $x^2 + x - 12 = 0$ was difficult to factorise, then the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

could have been used to find the $x$-intercepts.
Finding the axis of symmetry
Since the parabola \( y = x^2 + x - 12 \) has a vertical axis of symmetry, the axis of symmetry will cut the \( x \)-axis half-way between \((-4, 0)\) and \((3, 0)\), which are the two \( x \)-intercepts. The axis of symmetry will have the equation \( x = \frac{-4 + 3}{2} \), i.e. \( x = -\frac{1}{2} \) is the axis of symmetry.

For the parabola \( y = x^2 + x - 12 \),
\[ a = 1, \quad b = 1, \quad c = -12. \]
\[ \therefore \text{The axis of symmetry is} \quad x = \frac{-1}{2(1)} \]
\[ \text{i.e.} \quad x = -\frac{1}{2} \]

Finding the vertex (or turning point) and the maximum or minimum value
As the vertex lies on the axis of symmetry, its \( x \)-coordinate will be the same as that of the axis of symmetry. The \( y \)-coordinate can be found by substituting this \( x \) value into the equation of the parabola.
For \( y = x^2 + x - 12 \), the axis of symmetry is \( x = -\frac{1}{2} \).

Now, when \( x = -\frac{1}{2} \),
\[ y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12 \]
\[ y = -12\frac{1}{4} \]
\[ \therefore \text{The vertex of the parabola is} \quad (-\frac{1}{2}, -12\frac{1}{4}). \]
The minimum or maximum value of \( y \) will occur at the vertex. The parabola will have:

- a minimum value of \( y \) if the parabola is concave up (when the coefficient of \( x^2 \) is positive, eg \( y = 2x^2 \))
- a maximum value of \( y \) if the parabola is concave down (when the coefficient of \( x^2 \) is negative, eg \( y = -2x^2 \))

Hence, on the parabola \( y = x^2 + x - 12 \), the minimum value of \( y \) is \(-12\frac{1}{4}\) when \( x = -\frac{1}{2} \).

The method of completing the square can also be used to find the minimum or maximum value and the vertex as shown below.

\[
y = x^2 + x - 12 \\
y = (x^2 + x + \frac{1}{4}) - \frac{1}{4} - 12 \\
y = (x + \frac{1}{2})^2 - 12\frac{1}{4}
\]

As \((x + \frac{1}{2})^2\) is always greater than or equal to 0, the minimum value of \( y \) will be \(-12\frac{1}{4}\) when \( x = -\frac{1}{2} \) and the vertex is the point \((-\frac{1}{2}, -12\frac{1}{4})\).

### Worked examples

For each equation, find:

- a the \( y \)-intercept
- b the \( x \)-intercepts
- c the axis of symmetry
- d the vertex (turning point)

Use these results to sketch each graph.

1. \( y = x^2 + 2x - 3 \)
2. \( y = 2x^2 + 4x + 3 \)
3. \( y = 4x - x^2 \)

#### Solutions

1. **a** For the \( y \)-intercept, let \( x = 0 \)
   \[
y = (0)^2 + 2(0) - 3 \\
The \( y \)-intercept is \(-3\).
   
   **b** For the \( x \)-intercepts, let \( y = 0 \)
   \[
   0 = x^2 + 2x - 3 \\
   (x + 3)(x - 1) = 0 \\
The \( x \)-intercepts are \(-3\) and \(1\).
   
   **c** Axis of symmetry:
   \[
x = \frac{-3 + 1}{2} \quad \text{(midpoint of \( x \)-intercepts)} \\
\therefore \ x = -1 \text{ is the axis of symmetry.}
   
   **d** To find the vertex, substitute \( x = -1 \) into the equation to find the \( y \)-value.
   \[
y = (-1)^2 + 2(-1) - 3 \\
y = -4 \\
\therefore \ 	ext{The vertex is \((-1, -4)\).}
   
   We now plot the above information on a number plane and fit the parabola to it.
To find the $y$-intercept of 
$y = 2x^2 + 4x + 3$, let $x$ be zero.

$y = 2(0)^2 + 4(0) + 3$

∴ The $y$-intercept is 3.

For the $x$-intercepts, solve

$2x^2 + 4x + 3 = 0$. However, when
we use the formula, we get

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

where $a = 2, b = 4, c = 3$.

$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 2 \times 3}}{2 \times 2}$

$x = \frac{-4 \pm \sqrt{-8}}{4}$

This gives us a negative number
under the square root sign.

You can't find the square root of
a negative number!

Thus, there are no solutions, so the
parabola does not cut the $x$-axis.

$y = 2x^2 + 4x + 3$ has $a = 2, b = 4, c = 3$.

∴ Axis of symmetry is $x = \frac{-b}{2a}$.

ie $x = \frac{-4}{2(2)}$

∴ The axis of symmetry is $x = -1$.

The vertex is the turning point of
the curve, and is on the axis of
symmetry $x = -1$.

When $x = -1$,

$y = 2(-1)^2 + 4(-1) + 3$

$= 1$

∴ The vertex is $(-1, 1)$.
Exercise 5:02

For each of the graphs, find:

i the y-intercept

ii the x-intercepts

iii the equation of the axis of symmetry

iv the coordinates of the vertex

1 a For the parabola $y = ax^2 + bx + c$:

\[ y = 4x - x^2 \]

- If $x = 0$, then $y = 0$.
  \[
  \therefore \text{ The curve cuts the y-axis at the origin.}
  \]

b When $y = 0$,

\[ 4x - x^2 = 0 \]

\[ x(4 - x) = 0 \]

\[ \therefore \text{ The x-intercepts are 0 and 4.} \]

c Axis: $x = 2$ (midpoint of x-intercepts)

d When $x = 2$,

\[ y = 4(2) - (2)^2 \]

\[ = 4 \]

\[ \therefore \text{ The vertex is } (2, 4). \]

This is a ‘sad’ graph because the coefficient of $x^2$ is negative.
2 Find the y-intercepts of the following parabolas.
   a \( y = x^2 - 6x + 5 \)
   b \( y = 2x^2 - 8 \)
   c \( y = (x - 2)(x + 3) \)

3 Find the x-intercepts of the following parabolas.
   a \( y = x^2 - 2x - 8 \)
   b \( y = 3x^2 + 10x - 8 \)
   c \( y = (x - 3)(4x + 7) \)

4 Find the equation of the axis of symmetry and the coordinates of the vertex of the following parabolas.
   a \( y = (x - 3)(x - 5) \)
   b \( y = 3(x - 2)(x + 6) \)
   c \( y = \frac{-1}{2}(x + 4)(2 - x) \)
   d \( y = x^2 - 6x + 7 \)
   e \( y = 3x^2 - 9x + 14 \)
   f \( y = 4 - 3x - x^2 \)

5 Find the minimum value of \( y \) on the following parabolas.
   a \( y = x^2 - 6x - 2 \)
   b \( y = 4x^2 - 4x + 6 \)
   c \( y = 9x^2 - 30x + 18 \)

6 Find the maximum value of \( y \) on the following parabolas.
   a \( y = 1 - 2x - x^2 \)
   b \( y = -4x^2 + 20x - 27 \)
   c \( y = 7 - 12x - 9x^2 \)

7 For the parabola \( y = x^2 + 2x - 8 \), find:
   a the y-intercept
   b the x-intercepts
   c the axis of symmetry
   d the vertex
   e hence, sketch its graph

When finding the x-intercepts, if you can’t factorise, then use the formula.

8 Repeat the steps in question 7 to graph the following equations, showing all the relevant features.
   a \( y = x^2 - 6x + 5 \)
   b \( y = x^2 - 6x \)
   c \( y = 2x^2 - 8x - 10 \)
   d \( y = -x^2 + 4x - 3 \)
   e \( y = -x^2 + 6x - 9 \)
   f \( y = 2x^2 + 4x + 2 \)
   g \( y = x^2 - 3x - 4 \)
   h \( y = 2x^2 - 3x - 2 \)
   i \( y = -2x^2 - 3x - 1 \)

9 Match each graph with one of the equations written below the diagram. Each graph has an \( x^2 \) shape.

   a \( y = -x^2 + 3 \)
   b \( y = x^2 - 2x + 2 \)
   c \( y = x^2 - 8x + 12 \)
   d \( y = x^2 + 8x + 16 \)
   e \( y = -x^2 - 6x - 10 \)
   f \( y = x^2 - 8x + 18 \)
Sketch each set of three parabolas on the same number plane.

a i \( y = x^2 - 4 \)  
ii \( y = x^2 - 4x \)  
iii \( y = x^2 - 4x + 4 \)

b i \( y = 9 - x^2 \)  
ii \( y = 9x - x^2 \)  
iii \( y = 10 + 9x - x^2 \)

c i \( y = (x - 3)(x + 5) \)  
ii \( y = 2(x - 3)(x + 5) \)  
iii \( y = (3 - x)(5 + x) \)

d i \( y = x^2 - 2x - 8 \)  
ii \( y = 2x^2 - 4x - 16 \)  
iii \( y = 8 + 2x - x^2 \)

Sketch the graph of each quadratic relationship, showing all relevant features.

a i \( y = 2x^2 - 8 \)  
b i \( y = 16 - x^2 \)  
c i \( y = (x + 2)(x - 6) \)

d i \( y = x^2 + 4x + 3 \)  
e i \( y = x^2 - 8x + 7 \)  
f i \( y = x^2 - 5x \)

g i \( y = (3 - x)(7 + x) \)  
h i \( y = 24 - 2x - x^2 \)  
i i \( y = 4x^2 + 16x + 7 \)

j i \( y = 2x^2 + 9x - 5 \)  
k i \( y = 4x^2 - 36x + 56 \)  
l i \( y = 2x^2 - 5x - 7 \)

The parabola in the diagram has its vertex at \((-1, -8)\) and it passes through the point \((1, 4)\).

The equation of the parabola has the form \( y = ax^2 + bx + c \).

a Use the y-intercept to show that \( c = -5 \).

b Use the equation of the axis of symmetry to show that \( b = 2a \) and that the equation of the parabola is of the form \( y = ax^2 + 2ax - 5 \).

c Substitute the coordinates of the vertex or the point \((1, 4)\) to find the value of \( a \).

d What is the equation of the parabola?

Use the method of question 13 to find the equation of each of the following parabolas.

a

b

c
Work out the answer to each question and put the letter for that part in the box that is above the correct answer.

**Factorise:**

E: \( x^2 - 3x - 4 \)  
S: \( x^2 - 16 \)  
O: \( x^2 - 4x \)  
S: \( x^2 + 3x - 4 \)

**What is the axis of symmetry for:**

H: \( y = x^2 - 4x + 4 \)  
S: \( y = x^2 + 4x \)  
K: \( y = x^2 - 4 \)  
O: \( y = x^2 - 3x - 4 \)

**Where does each parabola below cut the y-axis?**

I: \( y = x^2 - 4x + 4 \)  
H: \( y = x^2 - 4x \)  
T: \( y = x^2 - 4 \)

**What is the vertex for each parabola?**

L: \( y = x^2 - 4x + 4 \)  
C: \( y = x^2 + 4x \)  
L: \( y = x^2 + 2x - 3 \)
We need to take many points when graphing a curve like \( y = \frac{2}{x} \), as it has two separate parts. The curve of such an equation is called a hyperbola.

\[
y = \frac{k}{x}
\]

is a hyperbola if \( k \) is a constant (eg 1, 2 or 4).

### Prep Quiz 5:03

Find the value of \( \frac{2}{x} \) when \( x \) is:  
1 \( \frac{1}{2} \)  
2 1  
3 2  
4 4  
5 \( -4 \)

If \( y = \frac{8}{x} \), what is the value of \( y \) when \( x \) is:  
6 \( 2? \)  
7 \( 4? \)  
8 \( -8? \)

If \( y = \frac{4}{x} \), what happens to \( y \) as \( x \) increases from 1 to 40?

If \( y = \frac{4}{x} \), what happens to \( y \) as \( x \) decreases from \( -1 \) to \( -40 \)?

### Worked example

\[
y = \frac{2}{x}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(-0.5)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(-0.5)</td>
<td>(-0.7)</td>
<td>(-1)</td>
<td>(-2)</td>
<td>(-4)</td>
<td>(-)</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0.7</td>
<td>0.5</td>
</tr>
</tbody>
</table>

- Notice that there is no value for \( y \) when \( x = 0 \).
  - When \( x = 0 \), \( y = \frac{2}{x} \) becomes \( y = \frac{2}{0} \). This value cannot exist as no number can be divided by 0.
- What will happen to the \( y \) values as the \( x \) values get closer to 0?
- What will happen to the \( y \) values as the \( x \) values become larger?
Chapter 5 Number Plane Graphs and Coordinate Geometry

Use your calculator to complete the tables below, giving values for $y$ correct to two decimal places.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-2</th>
<th>-1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-2</th>
<th>-1</th>
<th>1</th>
<th>2</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>-4</th>
<th>-2</th>
<th>-1</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Complete the table below for $y = \frac{4}{x}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-8</th>
<th>-4</th>
<th>-2</th>
<th>-1</th>
<th>-0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Use a sheet of graph paper to graph the curve $y = \frac{4}{x}$ using your table. Use values $-8$ to $8$ on both axes.

There seems to be a pattern here.

Note:
- The hyperbola has two parts.
- The parts are in opposite quadrants and are the same shape and size.
- The curve is symmetrical.
- The curve approaches the axes but will never touch them.
- The $x$- and $y$-axes are called asymptotes of the curve.
- No value for $y$ exists when $x = 0$. 

Exercise 5:03
Graph the curve \( y = -\frac{1}{x} \) by first completing the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-2</th>
<th>-1</th>
<th>-0.5</th>
<th>-0.25</th>
<th>0.25</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y  )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What does a negative value of \( k \) do to the graph?

Match each of the graphs A to F with the following equations.

- \( a \) \( y = \frac{4}{x} \)
- \( b \) \( y = -\frac{4}{x} \)
- \( c \) \( y = \frac{20}{x} \)
- \( d \) \( y = \frac{10}{x} \)
- \( e \) \( y = -\frac{12}{x} \)
- \( f \) \( y = \frac{16}{x} \)

Does the point (4, 2) lie on the hyperbola \( y = \frac{8}{x} \)?

If the point (3, -6) lies on the hyperbola \( y = \frac{k}{x} \), what is the value of \( k \)?

The hyperbola \( y = \frac{k}{x} \) passes through the point (10, 2). What is the value of \( k \)?

For each of the following, find a point that the hyperbola passes through and, by substituting this in the equation \( y = \frac{k}{x} \), find the equation of the hyperbola.

- \( a \)
- \( b \)
Chapter 5 Number Plane Graphs and Coordinate Geometry

5:04 Exponential Graphs: $y = a^x$  Outcome PAS5-3-4

Prep Quiz 5:04

Find the value of:  

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^3$</td>
<td>$2^5$</td>
<td>$2^0$</td>
<td>$2^{-1}$</td>
<td>$2^{-5}$</td>
<td></td>
</tr>
</tbody>
</table>

If $y = 2^{-x}$, find $y$ when $x$ is:  

<table>
<thead>
<tr>
<th></th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>−2</td>
<td></td>
</tr>
</tbody>
</table>

Use your calculator to find, to one decimal place, the value of:  

<table>
<thead>
<tr>
<th></th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{1.5}$</td>
<td>$2^{-2.5}$</td>
<td></td>
</tr>
</tbody>
</table>

A curve whose equation is of the form $y = a^x$ is called an exponential curve. On the following number plane, the graph of $y = 2^x$ has been drawn.

- The curve passes through (0, 1) on the $y$-axis since $2^0 = 1$.
- The curve rises steeply for positive values of $x$.
- The curve flattens out for negative values of $x$. The $x$-axis is an asymptote for this part of the curve.
- Because $2^x$ is always positive, the curve is totally above the $x$-axis.
Exercise 5:04

1. a Complete the table below for \( y = 2^x \) and graph the curve for \(-2 \leq x \leq 3\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1.6</th>
<th>-1.2</th>
<th>-0.8</th>
<th>-0.4</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0.57</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(In the table, use values of \( y \) correct to two significant figures.)

b Complete the table below for \( y = 3^x \) and graph the curve for \(-2 \leq x \leq 2\). Use the same diagram you used in part a.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c Compare the graphs of \( y = 2^x \) and \( y = 3^x \). What do you notice?

2. a Complete the table of values for \( y = 2^{-x} \) and graph the curve on a number plane.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b Compare your graph with \( y = 2^x \). What do you notice?

3. a Draw on the same number plane the graphs of \( y = 2^x \) and \( y = 2^{-x} \).

b With reference to the graphs in part a, now draw the graphs of \( y = -2^x \) and \( y = -2^{-x} \) on the same diagram.

c What is the effect of graphing the ‘negative’ relationships?

4. The graphs of the curves \( y = 2^x, y = 2 \times 2^x, y = 3 \times 2^x \) and \( y = 0.5 \times 2^x \) are shown on the number plane.

a Match each of the curves A to D with its equation.

b What is the effect of multiplying an exponential function by a constant, \( k \) (ie \( y = ka^x \))?
a For the graph of \( y = 6a^x \), where would the curve cut the \( y \)-axis? To which end of the \( x \)-axis is the curve asymptotic? (Note: \( a > 0 \))

b For the graph of \( y = 6a^{-x} \), where would the curve cut the \( y \)-axis? To which end of the \( x \)-axis is the curve asymptotic?

c Using your answers to parts a and b, draw sketches of:

i \( y = 4 \times 2^x \)  
ii \( y = 2 \times 2^{-x} \)  
iii \( y = \frac{1}{2} \times 3^x \)  
iv \( y = 5 \times 4^{-x} \)

6 The quantity of carbon-14 present after \( t \) years is given by the formula:

\[
Q = A \times 2^{\frac{-t}{5730}}
\]

where \( Q \) is the quantity of carbon-14 present, \( A \) was the amount of carbon-14 present at the start, and \( t \) is the time in years.

If 10 g of carbon-14 were present at the start, \( A = 10 \), and the formula becomes:

\[
Q = 10 \times 2^{\frac{-t}{5730}}
\]

a Find the value of \( Q \) when \( t = 0 \).

b Find the quantity of carbon-14 remaining after 5730 years. (Carbon-14 has a half-life of 5730 years.)

c Find the value of \( Q \) when \( t \) is:

i 11 460  
ii 17 190  
iii 2865

d Use the values found above to sketch the graph of \( Q = 10 \times 2^{\frac{-t}{5730}} \) for values of \( t \) from 0 to 20 000.

Fun Spot 5:04 The tower of Hanoi

This famous puzzle consists of three vertical sticks and a series of discs of different radii which are placed on one stick to form a tower, as shown in the diagram. The aim of the puzzle is to move the discs so that the tower is on one of the other sticks. The rules are:

• only one disc can be moved at a time to another stick  
• a larger disc can never be placed on top of a smaller one.

The puzzle can be made more difficult by having more discs.

Investigate the minimum number of moves needed if there are 2, 3 or 4 discs. Can you generalise your results?

Can you predict the minimum number of moves needed if there are, say, 8 discs? (Hint: An exponential relationship can be found!)
5:05 The Circle

A circle may be defined as the set of all points that are equidistant (the same distance) from a fixed point called the centre.

- We need to find the equation of a circle of radius \( r \) units with the origin \( O \) as its centre.
- If \( P(x, y) \) is a point on the circle which is always \( r \) units from \( O \), then, using Pythagoras’ theorem, \( x^2 + y^2 = r^2 \). This is the equation that describes all the points on the circle.

**Worked examples**

1. What is the equation of the circle that has its centre at the origin and a radius of 6 units?
2. What is the radius of the circle \( x^2 + y^2 = 5 \)?

**Solutions**

1. \( r = 6 \), so the equation is \( x^2 + y^2 = 6^2 \)
   
   \[ x^2 + y^2 = 36 \]

   \( \therefore \) \( x^2 + y^2 = 36 \) is the equation of the circle.

2. \( x^2 + y^2 = 5 \) is of the form \( x^2 + y^2 = r^2 \).

   \[ r^2 = 5 \]

   \[ \therefore \] The radius of the circle is \( \sqrt{5} \) units.

**Exercise 5:05**

1. What is the equation of each circle?

   a. ![Circle](image)

   b. ![Circle](image)

   c. ![Circle](image)

   **Foundation Worksheet 5:05**

   The circle PAS5·3·4
   
   1. For each circle, write down its
      
      i. radius
      ii. equation
   
      a. ![Circle](image)
      b. ![Circle](image)
   
   2. Sketch the circle represented by the equations:
      
      a. \( x^2 + y^2 = 36 \)
      b. \( x^2 + y^2 = 4 \)
2. What is the equation of a circle with the origin as its centre if the radius is:
   
   \( \text{a} \) 2 units? \( \text{b} \) 7 units? \( \text{c} \) 10 units?
   
   \( \text{d} \) \( \sqrt{3} \) units? \( \text{e} \) \( \sqrt{6} \) units? \( \text{f} \) 2\( \sqrt{2} \) units?
   
   \( \text{g} \) 1\( \frac{1}{2} \) units? \( \text{h} \) 2\( \frac{1}{4} \) units? \( \text{i} \) 4\( \cdot \)2 units?

3. What is the radius of these circles?
   
   \( \text{a} \) \( x^2 + y^2 = 64 \) \( \text{b} \) \( x^2 + y^2 = 81 \) \( \text{c} \) \( x^2 + y^2 = 2 \) \( \text{d} \) \( x^2 + y^2 = 6 \cdot 25 \) \( \text{e} \) \( x^2 + y^2 = 2 \) \( \text{f} \) \( x^2 + y^2 = 2 \) \( \text{g} \) \( 4x^2 + 4y^2 = 9 \) \( \text{h} \) \( 9x^2 + 9y^2 = 16 \)

4. For the circle \( x^2 + y^2 = 25 \), there are two points that have an \( x \) value of 3. Substituting \( x = 3 \) into \( x^2 + y^2 = 25 \)
   we get \( 3^2 + y^2 = 25 \).
   \( \therefore y^2 = 25 - 9 \)
   \( = 16 \)
   \( y = \pm 4 \) \([+4 \text{ or } -4]\)
   So (3, 4) and (3, -4) are the two points.

5. a Find the two points on the circle \( x^2 + y^2 = 25 \) that have an \( x \) value of:
   i 4 ii -3 iii 2
   
   b Find the two points on the circle \( x^2 + y^2 = 25 \) that have a \( y \) value of:
   i 4 ii -3 iii 2

6. Which equation in each part represents a circle?
   
   \( \text{a} \) \( y = 3x - 1 \), \( x^2 + 2x = y \), \( x^2 + y^2 = 1 \)
   
   \( \text{b} \) \( xy = 9 \), \( x^2 + y^2 = 9 \), \( x + y = 3 \)
   
   \( \text{c} \) \( x^2 = 4 - y^2 \), \( x^2 = y^2 - 4 \), \( x^2 = y + 4 \)
   
   \( \text{d} \) \( y^2 = 2x + x^2 \), \( y^2 = 2x^2 + 7 \), \( y^2 = 2 - x^2 \)

7. a How could it be determined whether a point was inside, outside or lying on a particular circle?
   
   b State whether these points are inside, outside or on the circle \( x^2 + y^2 = 20 \).
   i (2, 4) ii (4, 3) iii (-3, 3) iv (1, -4)
   v (-3, 4) vi (-4, 2) vii (2\( \frac{1}{2} \), 3\( \frac{1}{2} \)) viii (1-5, 4-5)

8. Find the equation of the circle with its centre at the origin that passes through the point:
   
   \( \text{a} \) (-4, 3) \( \text{b} \) (-2, -3) \( \text{c} \) (1, \( \sqrt{3} \))
Curves of the form $y = ax^3 + d$ are called **cubics** because of the $x^3$ term. The simplest cubic graph is $y = x^3$, which occurs when $a = 1$ and $d = 0$.

As with other graphs, a table of values is used to produce the points on the curve.

$$y = x^3$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-8</td>
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<td>-0.1</td>
<td>0</td>
<td>0.1</td>
<td>1</td>
<td>3.4</td>
<td>8</td>
</tr>
</tbody>
</table>

Features of $y = x^3$:
- It is an increasing curve.
- As $x$ increases, the value of $x^3$, and hence $y$, becomes large very quickly. This means it is difficult to fit the points on a graph.
- When $x$ is positive, $x^3$, and hence $y$, is positive.
  - When $x$ is negative, $x^3$, and hence $y$, is negative.
  - When $x$ is zero, $x^3$ is zero.

In this section, the relationship between the curve $y = x^3$ and the curve $y = ax^3 + d$ for various values of $a$ and $d$ will be investigated.

**Exercise 5:06**

1. a Match each of the equations below with the graphs A, B and C.
   - i $y = 2x^3$
   - ii $y = \frac{1}{2}x^3$
   - iii $y = 3x^3$

2. b Which graph increases the fastest? (Which is the steepest?)

3. c Which graph increases the slowest?

4. d How can you tell which graph is the steepest by looking just at the equations?
2 Which of the curves is steeper:
   a  \( y = x^3 \) or \( y = 3x^3 \)?
   b  \( y = x^3 \) or \( y = \frac{1}{2}x^3 \)?
   c  \( y = 2x^3 \) or \( y = 3x^3 \)?

3 The graphs of \( y = \frac{1}{2}x^3 \) and \( y = -\frac{1}{2}x^3 \) are shown.
   a  How are the graphs related?
   b  What is the effect on \( y = ax^3 \) of the sign of \( a \)?

4 From your results so far, you should have noticed that all the curves are either decreasing or increasing. Without sketching, state whether the following are increasing or decreasing.
   a  \( y = 4x^3 \)
   b  \( y = -10x^3 \)
   c  \( y = 0.25x^3 \)
   d  \( y = \frac{1}{5}x^3 \)
   e  \( y = -\frac{1}{5}x^3 \)
   f  \( y = -\frac{x^3}{5} \)

5 a Copy and complete the tables of values for the three curves \( y = x^3 \), \( y = x^3 + 2 \) and \( y = x^3 - 2 \).

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( x^3 )</th>
<th>( x^3 + 2 )</th>
<th>( x^3 - 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2</td>
<td>-8</td>
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</tr>
<tr>
<td></td>
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<td>-1</td>
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<td>1</td>
</tr>
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<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

b What is the equation of curves A and B?
c How is the graph of \( y = x^3 + 2 \) related to the graph of \( y = x^3 \)?
d How is the graph of \( y = x^3 - 2 \) related to the graph of \( y = x^3 \)?

6 Given the graph of \( y = \frac{1}{2}x^3 \), describe how you would obtain the graphs of:
   a  \( y = \frac{1}{2}x^3 + 1 \)
   b  \( y = \frac{1}{2}x^3 - 1 \)
   c  \( y = \frac{1}{2}x^3 + 2 \)
   d  \( y = \frac{1}{2}x^3 - 2 \)
   e  \( y = -\frac{1}{2}x^3 \)
   f  \( y = -\frac{1}{2}x^3 + 1 \)
In each diagram, the two curves A and B were obtained by moving the other curve up or down. Give the equations of the curves A and B.

![Diagram A](image1)

![Diagram B](image2)

![Diagram C](image3)

Sketch each pair of graphs on the same number plane.

- **a** \( y = 2x^3 \) \( y = 2x^3 - 2 \)
- **b** \( y = -x^3 \) \( y = -x^3 + 2 \)
- **c** \( y = \frac{1}{4}x^3 \) \( y = \frac{1}{4}x^3 + 4 \)

Sketch each pair of graphs on the same number plane.

- **a** \( y = x^3 + 1 \) \( y = -x^3 + 1 \)
- **b** \( y = 2x^3 + 1 \) \( y = x^3 + 1 \)

From the list of equations, write the letter or letters corresponding to the equations of the curves:

- **a** that can be obtained by moving \( y = x^3 \) up or down
- **b** that are the same shape as A
- **c** that are decreasing
- **d** that pass through \((0, 0)\)
- **e** that can be obtained from the curve \( y = -\frac{1}{3}x^3 \) by reflection in the y-axis
- **f** that have the largest y-intercepts

Give that each of the graphs is of the form \( y = ax^3 + d \), find its equation.

- **a** \( (2, 20) \)
- **b** \( (-2, 12) \)

For equations of the form \( y = ax^3 + d \), describe the effect on the graph of different values of \( a \) and \( d \).
Work out the answer to each question and put the letter for that part in the box that is above the correct answer.

For the number plane shown, match each graph with its correct equation below.

What is the equation of the parabola that results if the parabola \( y = x^2 \) is:
- \( O \) moved up 3 units
- \( A \) moved down 3 units
- \( E \) moved 3 units to the right
- \( R \) moved 3 units to the left
- \( R \) turned upside down and moved 3 units up
- \( O \) moved 3 units to the right and turned upside down

From the equations \( y = 2x \), \( y = x^2 \) and \( y = \frac{2}{x} \), which one is a:
- \( T \) parabola?
- \( L \) straight line?
- \( W \) hyperbola?

<table>
<thead>
<tr>
<th>Equation</th>
<th>( T )</th>
<th>( L )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = -1 )</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>( x = \frac{1}{(x-3)^2} )</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>( y = \frac{1}{x} )</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>( y = \frac{1}{x} )</td>
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<td>✔</td>
</tr>
<tr>
<td>( y = \frac{1}{x} )</td>
<td>✔</td>
<td>✔</td>
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<tr>
<td>( y = \frac{1}{x} )</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
</tbody>
</table>
### 5:07 Miscellaneous Graphs

It is important that you be able to identify the different graphs you have met so far by their equations. Study the review table below and then attempt the following exercise.

<table>
<thead>
<tr>
<th>Type of graph</th>
<th>Equation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight line</td>
<td>( y = mx + b ) or ( ax + by + c = 0 )</td>
<td><img src="image1" alt="Graph of a straight line" /></td>
</tr>
<tr>
<td>Lines parallel to the axes</td>
<td>( x = a ) or ( y = b )</td>
<td><img src="image2" alt="Graph of lines parallel to the axes" /></td>
</tr>
<tr>
<td>Parabola</td>
<td>( y = x^2 ) or ( y = ax^2 + bx + c )</td>
<td><img src="image3" alt="Graph of a parabola" /></td>
</tr>
<tr>
<td>Circle</td>
<td>( x^2 + y^2 = r^2 )</td>
<td><img src="image4" alt="Graph of a circle" /></td>
</tr>
<tr>
<td>Hyperbola</td>
<td>( y = \frac{k}{x} ) or ( xy = k )</td>
<td><img src="image5" alt="Graph of a hyperbola" /></td>
</tr>
<tr>
<td>Exponential curve</td>
<td>( y = a^x ) or ( y = a^{-x} )</td>
<td><img src="image6" alt="Graph of an exponential curve" /></td>
</tr>
<tr>
<td>Cubic curve</td>
<td>( y = x^3 ) or ( y = ax^3 + d )</td>
<td><img src="image7" alt="Graph of a cubic curve" /></td>
</tr>
</tbody>
</table>
**Exercise 5:07**

1. From the list of equations given on the right, choose those that represent:
   - a straight line
   - a circle
   - a parabola
   - a hyperbola
   - an exponential curve
   - a cubic curve

2. Sketch the graphs of the following equations, showing where each one cuts the coordinate axes.
   - a  \( y = 2x - 1 \)
   - b  \( y = 6 - x \)
   - c  \( x + 3y = 6 \)
   - d  \( x = -1 \)
   - e  \( y = 3 \)
   - f  \( x = 5 \)
   - g  \( y = x^2 + 2 \)
   - h  \( y = x^2 - 4 \)
   - i  \( y = (x - 1)^2 \)
   - j  \( y = (x + 1)(x - 3) \)
   - k  \( y = x^2 + 4x - 5 \)
   - l  \( y = -x^2 + 2 \)
   - m  \( y = 1 - x^2 \)
   - n  \( y = -x^2 + 2 \)
   - o  \( y = 5 - 4x - x^2 \)
   - p  \( x^2 + y^2 = 4 \)
   - q  \( x^2 + y^2 = 100 \)
   - r  \( x^2 + y^2 = 2 \)
   - s  \( y = \frac{2}{x} \)
   - t  \( xy = 4 \)
   - u  \( y = -\frac{3}{x} \)
   - v  \( y = 4^x \)
   - w  \( y = 2^{-x} \)
   - x  \( y = -3^x \)
   - y  \( y = 3x^3 - 3 \)
   - z  \( y = -3x^3 + 3 \)
   - \( 2 \)
   - \( 2 \)
   - \( 2 \)
   - \( 2 \)

3. Match each graph with its equation from the given list.

   - a  \( xy = 9 \)
   - b  \( y = -x^3 + 4 \)
   - c  \( y = 5^x \)
   - d  \( x^2 + y^2 = 4 \)
   - e  \( 2x - y + 2 = 0 \)
   - f  \( y = x^3 - 4 \)
   - g  \( y = 3 + 2x - x^2 \)
   - h  \( y = x^2 - 4 \)
   - i  \( y = \frac{5}{x} \)
   - j  \( 2x + 4y = 3 \)
   - k  \( y = 3 \)
   - l  \( 2x - 4y = 3 \)
   - m  \( y = \frac{1}{3}x^3 - 1 \)
Determine the equation of each graph.
Chapter 5 Number Plane Graphs and Coordinate Geometry

5:08 Using Coordinate Geometry to Solve Problems

**Prep Quiz 5:08**

A(−1, 2) and B(3, −2) are shown on the diagram.

Find:
1. the length of AB
2. the slope of AB
3. the midpoint of AB
4. the y-intercept of AB
5. the equation of the line AB

What is the gradient of the line:
6. \( y = 1 - 3x \)?
7. \( 2x + y = 4 \)?
8. What is the equation of the line that passes through (−2, 4) with a slope of 2?

What is the gradient of a line that is:
9. parallel to the line \( y = 2x - 3 \)?
10. perpendicular to the line \( y = 2x - 3 \)?
In Year 9, coordinate geometry was used to investigate:

- the distance between two points
- the midpoint of an interval
- the gradient (or slope) of an interval
- the various equations of a straight line
- parallel and perpendicular lines.

These results can be used to investigate the properties of triangles and quadrilaterals as well as other types of geometrical problems.

The results are reviewed in Chapter 1.

### Worked examples

**Example 1**

A triangle is formed by the points O(0, 0), A(2, 3) and B(4, 0). E and F are the midpoints of the sides OB and AB. Show:

a that $\triangle OAB$ is isosceles

b that EF is parallel to OB

**Solution 1**

a $OA = \sqrt{(2-0)^2 + (3-0)^2} = \sqrt{4 + 9} = \sqrt{13}$

b $AB = \sqrt{(4-2)^2 + (0-3)^2} = \sqrt{4 + 9} = \sqrt{13}$

∴ $\triangle OAB$ is isosceles (two equal sides).

b Now, E is $(1, 1 \frac{1}{2})$ and F is $(3, 1 \frac{1}{2})$.

∴ EF is horizontal (E and F have same y-coordinates).

OB is horizontal.

∴ EF is parallel to OB.

**Example 2**

W($-3, 0$), X$(2, 2)$, Y$(4, 0)$ and Z($-1, 2$) are the vertices of a quadrilateral.

a Show that WXYZ is a parallelogram

b Show that the diagonals bisect each other.

**Solution 2**

a Calculating the slopes of the four sides gives the following.

Slope of WX = $\frac{2 - 0}{2 - (-3)} = \frac{2}{5}$

Slope of ZY = $\frac{0 - (-2)}{4 - (-1)} = \frac{2}{5}$

Slope of WZ = $\frac{0 - (-2)}{-3 - (-1)} = \frac{-1}{2}$

Slope of XY = $\frac{2 - 0}{2 - (-4)} = \frac{2}{6}$

∴ WX $\parallel$ ZY (equal slopes) and WZ $\parallel$ XY (equal slopes)

∴ WXYZ is a parallelogram (opposite sides are parallel).
a Show that the triangle formed by the points \( O(0, 0), A(3, 1) \) and \( B(1, 3) \) is isosceles.

b Show that the triangle formed by the points \( (0, 0), (1, 3) \) and \( (7, 1) \) is right-angled.

c Show that the triangle with vertices at \( (0, 0), (-2, 2) \) and \( (2, 2) \) is both right-angled and isosceles.

b Midpoint of \( XZ = \left( \frac{2 + (-1)}{2}, \frac{2 + (-2)}{2} \right) \) \( = \left( \frac{1}{2}, 0 \right) \)

Midpoint of \( WY = \left( \frac{1}{2}, 0 \right) \)

\( \therefore \left( \frac{1}{2}, 0 \right) \) is the midpoint of both diagonals.

\( \therefore \) The diagonals \( XZ \) and \( WY \) bisect each other.

Exercise 5:08

1 a Show that the triangle formed by the points \( O(0, 0), A(3, 1) \) and \( B(1, 3) \) is isosceles.

b Show that the triangle formed by the points \( (0, 0), (1, 3) \) and \( (7, 1) \) is right-angled.

c Show that the triangle with vertices at \( (0, 0), (-2, 2) \) and \( (2, 2) \) is both right-angled and isosceles.

2 a Show that the quadrilateral with vertices at \( A(0, 2), B(3, 0), C(0, -2) \) and \( D(-3, 0) \) is a rhombus.

b A quadrilateral is formed by joining the points \( O(0, 0), B(1, 2), C(5, 0) \) and \( D(4, -2) \).

Show that it is a rectangle.

c The points \( A(0, 2), B(2, 0), C(0, -2) \) and \( D(-2, 0) \) are joined to form a quadrilateral.

Show that it is a square.

3 a If \( OABC \) is a rectangle, what are the coordinates of \( B \)?

b Find the length of \( OB \) and \( AC \).

What property of a rectangle have you proved?

c Find the midpoint of \( OB \) and \( AC \).

What does your answer tell you about the diagonals of a rectangle?

4 The points \( A(0, 0), B(6, 4) \) and \( C(4, -2) \) form a triangle.

a Find the midpoints of \( AB \) and \( AC \).

b Find the slope of the line joining the midpoints in a.

What is the slope of \( BC \)?

d What do your answers to parts b and c tell you?

5 The points \( A(4, 0), B(4, 4), C(0, 4) \) and \( D(0, 0) \) form a square. Find the slopes of \( BD \) and \( AC \).

What does your result say about the diagonals \( BD \) and \( AC \)?
6 A right-angled triangle OAB is shown.
   a Find the coordinates of E, the midpoint of AB.
   b Find the length of OE.
   c Find the length of EA.
   d What can you say about the distance of E from O, A and B?

7 A triangle has its vertices at the points A(−1, 3), B(2, 4) and C(1, 1).
   a Show that the triangle is isosceles.
   b Find E, the midpoint of AC.
   c Find the slope of the line joining E to B.
   d Show that EB is perpendicular to AC.
   e Describe how you could find the area of ΔABC.

8 a Find the midpoints of OA and AB.
   b Find the length of the line joining the midpoints in a.
   c Show that your answer in b is half the length of OB.

9 ABCD is a quadrilateral.
   a Find the coordinates of the midpoints of each side.
   b Join the midpoints to form another quadrilateral. What type of quadrilateral do you think it is?
   c How could you prove your answer in b?

10 The points A(−2, 0), B(0, 4) and C(4, 0) form the vertices of an acute-angled triangle.
   a Find the equation of the perpendicular bisectors of the sides AB, BC and AC.
   b Find the point of intersection of the perpendicular bisectors of the sides AB and AC.
   c Show that the perpendicular bisector of the side BC passes through the point of intersection found in b.
A median is a line joining a vertex of a triangle to the midpoint of the opposite side.

a) Find the equations of the medians.

b) Find the point of intersection of two of the medians.

c) Show that the third median passes through the point of intersection of the other two.
**Literacy in Maths**

### Maths terms 5

**circle**
- The equation of a circle in the number plane with its centre at the origin is: \[ x^2 + y^2 = r^2 \]
  where \( r \) is the radius.

**cubic curve**
- A curve that contains an \( x^3 \) term as its highest power. In this chapter, the curve’s equation is of the form: \( y = ax^3 + d \)

**exponential curve**
- A curve with an equation of the form \( y = a^x \), where \( a > 0 \).

**equation**
- An algebraic statement that expresses the relationship between the \( x- \) and \( y- \)coordinates of every point \((x, y)\) on the curve.

**graph (of a curve)**
- The line that results when the points that satisfy a curve’s equation are plotted on a number plane.

**hyperbola**
- A curve with the equation \( y = \frac{k}{x} \) where \( k \) is a constant.

- It has two asymptotes (the \( x \)-axis and \( y \)-axis), which are lines that the curve approaches but never reaches.

**parabola**
- A curve with the equation \( y = ax^2 + bx + c \).

- The line of symmetry of the parabola is its axis of symmetry. The equation of the axis of symmetry is \( x = -\frac{b}{2a} \).
- Parabolas can be concave up or concave down.
- The highest (or lowest) value of \( y \) on the parabola is the maximum (or minimum) value.
- The point where the parabola turns around is its vertex (or turning point).

**\( x- \) and \( y- \)intercept(s)**
- The point(s) where a curve crosses the \( x- \) or \( y \)-axis.
Diagnostic Test 5  Number Plane Graphs and Coordinate Geometry

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate an area of weakness.
- Each weakness should be treated by going back to the section listed.
These questions can be used to assess all or parts of outcomes PAS5.1.2, PAS5.2.4 and PAS5.3.4.

<table>
<thead>
<tr>
<th>Question</th>
<th>Graphs/Equations</th>
<th>Section</th>
</tr>
</thead>
</table>
| 1        | a. $y = x^2$  
b. $y = x^2 - 4$  
c. $y = x^2 + 2$ | 5.01 |
| 2        | a. $y = 2x^2$  
b. $y = \frac{1}{2}x^2$  
c. $y = -x^2$ | 5.01 |
| 3        | a. $y = (x - 1)^2$  
b. $y = (x + 3)^2$  
c. $y = (x - 2)^2 + 1$ | 5.01 |
| 4        | a. $y = (x - 1)(x + 3)$  
b. $y = x^2 - 6x$  
c. $y = 8 - 2x - x^2$  
d. $y = 4x^2 + 8x - 5$ | 5.02 |
| 5        | Find the x-intercepts for each of the parabolas in question 4. | 5.02 |
| 6        | Find the equation of the axis of symmetry for each of the parabolas in question 4. | 5.02 |
| 7        | Find the vertex of each of the parabolas in question 4. | 5.02 |
| 8        | Sketch each of the parabolas in question 4. Also state the maximum or minimum value for each quadratic expression. | 5.02 |
| 9        | Determine the equation of each parabola. | 5.02 |
| 10       | Sketch the graphs of:  
a. $y = \frac{3}{x}$  
b. $y = -\frac{2}{x}$  
c. $xy = 6$ | 5.03 |
11 Sketch, on the same number plane, the graphs of:
   a  \( y = 2^x \)        b  \( y = 3^x \)        c  \( y = 2^{-x} \)

12 a  What is the equation of this circle?
   b  What is the equation of a circle that has its centre at the origin and a radius of 7 units?
   c  What is the radius of the circle \( x^2 + y^2 = 3 \)?

13 Sketch the graphs of:
   a  \( y = x^3 + 1 \)        b  \( y = 1 - x^3 \)        c  \( y = 2x^3 - 1 \)

14 From the list of equations on the right, which one represents:
   a  parabolas?
   b  straight lines?
   c  circles?
   d  hyperbolas?
   e  exponential graphs?
   f  cubic graphs?

15 Match each graph with its equation from the list.
   a
   b
   c
   d
   e
   f

\[ A \ y = x^2 - 2 \]
\[ B \ y = 2^x \]
\[ C \ xy = 1 \]
\[ D \ y = -2x^2 \]
\[ E \ x^2 + y^2 = 2 \]
\[ F \ y = 3 \]
\[ G \ xy = 9 \]
\[ H \ y = 3^x \]
\[ I \ y = 9 - x^2 \]
\[ J \ x^2 + y^2 = 9 \]
\[ K \ y = 9 - x^3 \]
\[ L \ y = \frac{3}{x} \]